**System of Differential Equations**

**System of Homogeneous Linear Differential Equations:** Consider the following system of differential equations,

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where the coefficients  are real constants.

This is called system of homogeneous linear differential equations.

In matrix form the above system can be written as





where  and .

The solution of this system is



where , , …,  have a continuous derivative on real interval.

In fact, 

**Fundamental set:** If are linearly independent solutions of the homogeneous vector differential equation , then the set is called the fundamental set of this equation on .

**Fundamental Matrix:** A matrix whose individual columns consist of a fundamental set of solutions of is called a fundamental matrix of it.

Consider , ,, form a fundamental set of solutions of defined as

, , ,

Then the square matrix

is called the fundamental matrix.

**Question-01:** Prove that there exists fundamental set of solutions of the homogeneous system , where is a continuous matrix function.

**Solution:** Given that

we define a special set of constant vectors as

, , ,

That is, for each , has ith component one and all other components zero.

Now let be the solutions of (1) that satisfy the condition

,

That is, , , ,

where is an arbitrary point of .

Note that these solutions exist and are unique.

Now

for all .

This implies that the solutions are linearly independent on . Thus, form a fundamental set of (1).

Hence there exists a fundamental set of solutions of the homogeneous linear differential equation , where is a continuous matrix function. (**Proved**)

**Question-02:** Prove that the solutions of form an n-dimensional linear space, where is an n-dimensional vector and is an matrix.

**Solution:** Given that

Since is an matrix, so all solutions of (1) contain n components. Therefore, every solution vector belongs to where is a linear space of dimension

Thus, the solution set .

Since is the trivial solution of (1), so .

Let Then and . For any scalar we have

Therefore, is a subspace of and hence itself is a linear (vector) space.

Now we shall prove that has the dimension To prove this we show that has a basis with vectors.

Let , , be the solutions of (1) with initial conditions

, , , .

where , , are usual basis of .

Since is an matrix, so the above solutions exist and are unique with

Now we prove that the solution vectors , , linearly independent and they generate (span) .

**Independent part:** Let , , are scalars such that

Replacing by we get

Hence the solutions are linearly independent.

**Generator part:**

Since the solutions , , are linearly independent , so any solution of (1) can be written as

We shall prove that, this representation is unique.

If possible let,

where , , are scalars.

From (2) and (3)

Since , , are linearly independent so,

Using these values in (3) we get

which is same as (2).

Hence the representation (2) is unique.

Thus, any solution can be expressed as a unique linear combination of solutions

, ,. That is , , generates

Therefore the solutions set is a basis of and such dim

Hence the set of all solutions forms a linear space of dimension (**Proved**)

**System of Non-homogeneous Linear Differential Equations:** Consider the following system of differential equations,

****

where the coefficients  are real constants.

This is called system of non- homogeneous linear differential equations.

In matrix form the above system can be written as





where , , .

**Question-03:** State and prove the variation of constant formula.

**Solution: Statement:** If  is a fundamental matrix of on , then

is the unique solution of

,

where  is a fundamental matrix satisfying  is an vector, is an matrix, is an vector on and .

**Proof:** Here the homogeneous and non-homogeneous systems are

and

If  is a fundamental matrix of (1), then the general solution of (1) is

where is an arbitrary rowed constant vector.

For variation of constants (parameters), we replace of (3) by , and so we have

We now determine so that (4) is a solution of (2) with the condition .

Differentiating (4) with respect to we get,

By putting the values of (4) and (5) in (2) we get,

Since  is a fundamental matrix of the homogeneous system (1), so and on and hence exists and unique.

Therefore (1) gives,

Putting this value in (6) we get,

, where is an identity matrix

By integrating from to we get,

Putting this in (4) we get,

If then (8) reduces to

. (**Proved**)

**Question-04:** Prove that the solution of the nonhomogeneous system , is , where is an continuous matrix and  is a fundamental matrix of the corresponding homogeneous system and 

**Solution:** The given non-homogeneous equation is

The corresponding homogeneous equation of (1) is

Given  is fundamental matrix of (2).

We shall prove that solution of (1) can be expressed as

with initial condition

Let

we know that if

1. is any solution of (1)
2. ,,, are fundamental sets of (2)
3.  is a fundamental matrix of (2) having as its individual columns

then

where are suitably chosen constants and is an arbitrary rowed constant vector.

Using (5) and (7) we get from (6)

Putting in (8) we get

Putting the value of in (8) we get

Equations (3) and (10) are same.

Thus, with the initial condition , the solution of (1) is

(**Proved**)

**Problem**

**Problem-01:** Find a fundamental set of solutions of the system of equations , where

and .

**Solution:** Given that

where

We shall find a solution of the form

where is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of is

The characteristic polynomial of is,

=

The characteristic equation of is

Now gives

For we get from (3)

Putting we get .

For we get from (3)

, since both equations are same.

Putting we get .

The solutions are

The Wronskian of the solutions is

.

Thus, are linearly independent solutions of the given system.

The fundamental set is

(**Ans**)

**Problem-02:** Solve the system: ,

**Solution:** Given that

where and .

The solution of (1) is

where is an eigen vector.

The characteristic matrix of *A* is

The characteristic equation is

Now gives

, where

For we get from (3)

Putting in (4) we get

The solution is

.

Let be another solution of (1).

Then (1) must be satisfied by

This will be true if

Putting and we get and

The solution is

.

Therefore the solutions of the given system are

and .

**Problem-03:** Compute a fundamental matrix for the system:

and solve it.

**Solution:** Given that

where and

We shall find a solution of the form

where is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of is

The characteristic polynomial of is,

=

=

=

=

The characteristic equation of is

Now gives

For we get from (3)

Here is a free variable.

Putting we get and .

Again for we get from (3)

Here is a free variable.

Putting we get and .

Again for we get from (3)

Here is a free variable.

Putting we get and .

The solutions are

The wronskian of the solutions is

.

Thus, are linearly independent solutions of the given system and hence form a fundamental matrix.

The fundamental matrix is

The general solution of the given system is

.

**Problem-04:** Find the fundamental matrix for where .

**Solution:** Given that

where and

We shall find a solution of the form

where is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of is

The characteristic polynomial of is,

=

=

=

=

The characteristic equation of is

Now gives

For we get from (3)

Here is a free variable.

Putting we get and .

Again for we get from (3)

Here is a free variable.

Putting we get and .

Again for we get from (3)

Here is a free variable.

Putting we get and .

The solutions are

The wronskian of the solutions is

.

Thus, are linearly independent solutions of the given system and hence form a fundamental matrix.

The fundamental matrix is

(**Ans**)

**Problem-05:** Compute a fundamental matrix for the system of linear differential equation

, , . Hence solve the system.

**Solution:** Given that

In matrix form, the given system can be written as

where and

We shall find a solution of the form

where is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of is

The characteristic polynomial of is,

=

=

The characteristic equation of is

Now gives

For we get from (3)

Solving these equations we get and

Putting we get

Again for we get from (3)

Here is a free variable.

Putting we get and .

Again for we get from (3)

Here is a free variable.

Putting we get and .

The solutions are

The Wronskian of the solutions is

.

Thus, are linearly independent solutions of the given system and hence form a fundamental matrix.

The fundamental matrix is

The general solution of the given system is

.

where , and are arbitrary constants.

**Problem-06:** Solve

**Solution:** Given that:

and

The general form of non-homogeneous system is

with condition

If is a fundamental matrix of

then the solution of (3) can be expressed as

Comparing (1) with (3) and (2) with (4) we have

For fundamental matrix of (5), let the solution of (5) is , where is an eigen vector.

The characteristic equation is

Now gives

, where

For we get from (8)

Putting in (9) we get

Let be another solution of (5).

Then (5) must be satisfied by

This will be true if

Putting and we get and

The solutions are linearly independent.

The fundamental matrix for (5) is,

Here

and

Now

and

By putting the values of (9) and (10) in (6) we get

**(Ans)**

**Problem-07:** Solve:

**Solution:** Given that

where

and

The general form of non-homogeneous system is

with condition

If is a fundamental matrix of

then the solution of (3) can be expressed as

Comparing (1) with (3) and (2) with (4) we have

For fundamental matrix of (5), let , where is an eigen vector.

The characteristic equation is

Now gives

, where

For we get from (8)

Putting in (9) we get

Let be another solution of (5).

Then (5) must be satisfied by

This will be true if

Putting and we get and

The solutions are linearly independent.

The fundamental matrix for (5) is,

Here

and

Now

and

By putting the values of (9) and (10) in (6) we get

(**Ans**)

**Question**-08**:** Solve:

**Solution:** Given that:

and

The general form of non-homogeneous system is

with condition

If is a fundamental matrix of

then the solution of (3) can be expressed as

Comparing (1) with (3) and (2) with (4) we have

For fundamental matrix of (5), let , where is an eigen vector.

The characteristic equation is

Now gives

, where

For we get from (8)

Putting in (9) we get

Let be another solution of (5).

Then (5) must be satisfied by

This will be true if

Putting and we get and

The solutions are linearly independent.

The fundamental matrix for (5) is,

Here

and

Now

and

By putting the values of (9) and (10) in (6) we get

**Question**-09**:** Solve

**Solution:** Given that:

and

The general form of non-homogeneous system is

with condition

If is a fundamental matrix of

then the solution of (3) can be expressed as

Comparing (1) with (3) and (2) with (4) we have

For fundamental matrix of (5), let , where is an eigen vector.

The characteristic equation is

Now gives

, where

For we get from (8)

Putting in (9) we get

For we get from (8)

Putting in (9) we get

The wronskian is

The solutions are linearly independent.

The fundamental matrix for (5) is,

Here

and

Now

and

By putting the values of (9) and (10) in (6) we get

**(Ans)**